Formation of Teams in Contests: Tradeoffs Between Inter- and Intra-Team Inequalities*

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Abstract

We consider a team contest in which players make efforts to compete with other teams for a prize, and players of the winning team divide the prize according to a prize-sharing rule. This prize-sharing rule matters in generating members' efforts and attracting players from outside. Assuming that players differ in their abilities to contribute to a team and their abilities are observable, we analyze which team structure realizes by allowing players to move across teams. This inter-team mobility is achieved via head-hunting: a team leader can offer one of the positions to an outside player. We say that it is a successful head-hunting if the player is better off by taking the position, and the team's winning probability is improved. A team structure is stable if there is no successful head-hunting opportunity. We show that if all teams employ the egalitarian sharing rule, then the complete sorting of players according to their abilities occurs, and inter-team inequality becomes the largest. In contrast, if all teams employ a substantially unequal sharing rule, there is a stable team structure with a small inter-team inequality and a large intra-team inequality. This result illustrates a trade-off between intra-team inequality and inter-team inequality in forming teams.

Keywords: team contest, CES effort aggregator function, prize sharing rule, head-hunting, stable team structure, intra-team inequality, inter-team inequality

***JEL* Classification Numbers**: C71, C72, C78, D71, D72, D74

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1 Introduction

In team contests players have to exert joint effort in order to compete with other teams for a prize. Each player's performance is typically affected by internal team characteristics such as the team composition (the ability or productivity of one's teammates), the agreed-upon prize sharing rule, the complementarity between individual efforts, as well as the presence of free-riding incentives. The resulting teammates' efforts are then aggregated to team effort which, when measured against that of other teams, determines the team's winning probability according to a Tullock contest success function. Thus, the relative strength and composition of opposing teams can affect not only the equilibrium effort choice of each player but the equilibrium team composition in the first place.

In this paper we focus on a specific type of player mobility across teams, namely mobility achieved through head-hunting. We assume that in an attempt to improve their equilibrium winning probability, teams can extend an offer to any player. This offer must specify the new recruit's relative position on the team, characterized by a predetermined value for their share of the prize. When players consider the decision of potentially accepting such an offer, they have to weigh multiple costs and benefits: (1) what is the share of the prize they would receive on the other team? (2) how much effort would they have to exert there? (3) what is the new team's winning probability, and how important would their new position be in affecting the team's performance? (4) how does their (publicly known) ability compare to that of their new teammates, and could a potential transfer lead to increased free-riding incentives? (5) what happens to the seat they vacated on their current team (if they had one), and how will this affect the competition as a whole? When a player finds an offer acceptable, and if their new team's equilibrium winning probability is increased as a result of them joining, then we say that a successful head-hunting occurs. A team structure (a matching between teams and players) that allows no successful head-hunting will be defined as head-hunting-proof (or stable).

The goal of this work is to study the types of stable team structures that might result from common reward allocation rules. We assume that all teams have the same fixed capacity and that they all use the same common prize allocation rule. This sharing rule might be imposed as a part of the rules of the contest (such as the Kaggle example below), or it might just be the result of a long-established social norm in each industry. We distinguish between different allocation rules according to how equally they treat team members. At one end of the spectrum we consider the egalitarian rule which divides the prize equally among the players. At the other end are rules that treat players unequally, giving higher-ranked members substantially higher shares. Before proceeding with an overview of our findings, we present several examples that help illustrate the use of such allocation rules and the resulting team structures.

The first example comes from the website Kaggle, a platform hosting a variety of data science and machine learning competitions. Each competition is self-contained, with a predetermined prize, and there is a global cap of eight players per team (although each competition often has a lower team size limit). Anyone is allowed to participate and submit a solution, and joining a team is left entirely in the hands of the contestants. Committing to a team occurs at the time of registration for each contest and results in a binding agreement. When a team wins a competition, the prize is equally divided between team members regardless of their contribution (the egalitarian rule is enforced by Kaggle). It should be noted that all solutions are evaluated and ranked, not only the winning one. Each player is then granted a score representing how well their team did in the contest. The score is publicly known and can help players nd teams in future contests. It has been observed that over time many of the teams have ended up sorted by ability - some of the highest-ranked contestants have joined teams together, winning or scoring high in multiple competitions. The same seems to be true at the middle- and lower-end of scores as well. The continued evolution of team formation at Kaggle is very reminiscent of head-hunting and has led to an outcome in which the resulting inter-team inequality stands in contrast to the implicitly enforced intra-team equality.

team output is the result of aggregated individual e ort inputs, which we model via a CES aggregator function, allowing for di erent levels of e ort complementarity. Players' e orts are *not observable* or *not contractable* | thus players' e orts contribute to the winning probability of their team but do not a ect their shares of the winning prize. The shares that players receive are allowed to be heterogeneous based on the positions they are assigned to, and we explicitly focus on the rate at which lower positions are discounted relative to higher positions within each team. It should be noted that by combining the approaches by Konishi and Pan (2020, 2021) and Simeonov (2020), we can explicitly solve for player's equilibrium payo s, which makes it possible to discuss head-hunting as a well-de ned process of attracting better candidates.

The main result of this work is to show that the tradeo s between intra- and inter-team inequalities are not coincidental. We show that when the egalitarian rule is used within each team, then complete ability sorting across teams is the only stable team structure. Alternatively, we consider hierarchical prize allocation rules in which a common discount factor for rewards is used. For high discount factors, we show that the cyclical allocation of players across teams is stable. For intermediate discount factors, both the cyclical and complete sorting by ability can coexist, and more generally, a combination of cyclical assignment and ability sorting can occur in a stable team structure.

Much of the rationale behind these results originates from our key Lemma 3 in Section 4 below. It would be instructive to diverge with a brief discussion of Lemma 3 before proceeding with the model. Consider in particular a scenario with two teams: a strong team A with high average team ability and a high equilibrium chance of winning and a weaker team B with lower average team ability and low chance of success. Suppose, however, that the strongpericalofev s,evev-367 teams? Clearly, there must be a significant difference in compensation between group members to open the possibility for such an occurrence. Only then would a high-ability player find it viable to join a higher position on a weaker team instead of keeping a lower position on a more successful team. High inequality within teams seems to become a necessary prerequisite for achieving a more even distribution of talent across teams.

The rest of the paper is organized as follows. The following section presents a brief review of related literature. Section 2 describes the model and assumptions. Section 3 presents the equilibrium player and team efforts in general team contests. In Section 4, we proceed with the discussion of stability and the main results regarding the tradeoffs between intra- and inter-team inequalities, and Section 5 concludes.

1.1 Relations to the Literature

Broadly, this paper belongs to the theory of coalition formation with externalities. Players' payoffs depend not only on which coalition they belong to but also on other coalitions. Hart and Kurz (1983), Bloch (1996), Yi (1997), Ray and Vohra (1999), and Ray (2008) provide a general analysis of coalition formation games with externalities across coalitions. As specific economic applications, Bloch (1995), Yi (1996), and Ray and Vohra (2001) consider cartel structures, customs unions, and public good provision groups, respectively. Our paper belongs to this literature, but there are some differences: in our game, there is a membership quota for each team, and prize-sharing rules within a team are predetermined, but shares can be heterogeneous. Thus, each position of a team can be heterogeneous for players, and players care about which position of a team they will be assigned to. This is a new feature of our model in the coalition formation literature.

More specifically, this paper belongs to the literature on group contests and prize-sharing rules. Assuming individual efforts are contractable, Nitzan (1991) analyzes how the combination of an egalitarian and a relative-effort-sharing rules affects members' incentives for players in large and small groups. Lee (1995) and Ueda (2002) endogeneize group sharing rules in this class. Esteban and Ray (2001) and Nitzan and Ueda (2011) show that Olson's (1973) group size paradox disappears if the prize among the members can be allocated into public and private benefits and if private benefits can be allocated by an endogenously chosen relative-effort-sharing rule, respectively. Based on the line of group contest research above, Baik and Lee (1997, 2001) endogenize the alliance formation in Nitzan's (1991) game with endogenous group sharing rules and analyze two- and multiple-alliance cases, respectively. They use open-membership games to describe alliance formation. Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games. Sanchez-Pages (2007a,b) explores different types of stability concepts in alliance formation in contests where efforts are perfect substitutes. These papers assume alliance members can write a binding contract of sharing rules in the case of the alliance's winning. In contrast, following Esteban and Sakovics (2003), Konishi and Pan (2020, 2021) analyze equilibrium alliance structures in homogenous player alliance formation games without side payments when members' e orts are complementary with each other by using a CES aggregator function.[2] The current paper extends Konishi and Pan (2020,2021), allowing for heterogeneous abilities and unequal sharing rules using the approach by Simeonov (2020) and Kobayashi, Konishi, and Ueda (2023).[3] Unlike in Nitzan (1991) and Nitzan and Ueda (2011), individual e orts are unobservable or noncontractable, allowing for free-riders as in Esteban and Ray (2001). For more complete surveys of the literature on group contests, see Konrad (2009) and Fu and Wu (2019).

Our stability notion, head-hunting-proofness, is close to pairwise stability in matching literature due to the presence of team membership quotas. Gale and Shapley (1962) introduce the celebrated two-sided matching problem and its solution concept, pairwise stable matching. In their domain, the pairwise stability is equivalent to the core despite its simplicity. Imamura, Konishi, and Pan (2021) introduce externalities across matched pairs to the two-sided matching problems and show that their pairwise stable matching via swapping preserves nice properties.

2 The Model

There are potentially $j = 1, 2, \ldots, J$ teams, and there are M positions in a team. Let (m, j) stand for the mth position in j team. Player $i = 1, \ldots, N$ is characterized by her *ability* a_i. We assume that $a_1 \geq a_2 \geq \ldots \geq a_N$. With some abuse of notations, we also let M, J, and N stand for the set of positions, teams, and players, respectively. A *membership profile* is $\varphi = (\varphi_{mj})_{m \in M, j \in J}$ where $\varphi_{mj} \in N \cup \{\emptyset\}$ for all $m \in M$ and $j \in J$. We assume a player can only belong to a team. Therefore, a membership profile is *feasible* if $\varphi_{mj} \neq \varphi_{m'j'}$ for all $(m, j) \neq (m', j')$. Let $N_j = \{i \in N | \varphi_{mj} = i$ for some $m \in M\} \subseteq N$ be the set of players in team j under φ.

We will consider our team stability problem in a team contest framework in two stages. In stage 1, a team structure φ is determined, and in stage 2, an actual team contest occurs given φ. Membership profile φ is formed in stage 1, by players' foreseeing the resulting outcomes in stage 2. So, we will first describe the team contest problem in stage 2, and our stability notion in stage 1 will be introduced in Section 4.[8]

Given a feasible membership profile φ, players compete with each other as a team for a prize, which value is V. In this contest, team members $i \in N_j$ choose their effort levels e_i simultaneously and non-cooperatively. The members' efforts in team j are aggregated by a CES function $X_j = (\sum_{m \in M} a^{\sigma}_{\varphi_{mj}} e^{\sigma}_{\varphi_{mj}})^{\frac{1}{\sigma}}$, where $0 < \sigma < 1$.[9] This CES aggregator function becomes a linear function (perfect substitutes) when $\sigma = 1$, and becomes a Cobb-Douglas function when $\sigma = 0$ in the limit. Teams' aggregate effort vector $(X_1, \ldots, X_J)$ determines each team's winning probability. The winning probabilities of teams are determined by a Tullock-style contest: team j's "winning probability" is given by

$$P_j = \frac{X_j}{\sum_{k=1}^{J} X_k}. \tag{1}$$

After the winning team gets the prize, it will distribute the prize to its team members by a *common fixed sharing rule* that is considered as a social norm. This common sharing rule is $\alpha = (\alpha_1, \ldots, \alpha_m, \ldots, \alpha_M)$ with $\alpha_m \in [0, 1]$ and $\sum_{m \in M} \alpha_m = 1$, in which α_m stands for the prize share that the player in position m of a team. Without loss of generality, we rank positions in a team by its shares, that is, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_M$. The effort cost function is common for all players: player i's effort is common and linear $c_i(e_i) = e_i$. Therefore, the expected payoff of the player in the position m of team j, or, equivalently, the player i such that φ_{mj}

is obtained. Thus, we have

1

and

$$P_j = 1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J} \frac{1}{A_k(\varphi_k)}};$$

where $A_j(\varphi_j) = \left(\sum_{m=1}^{M} a_{m}^{\frac{\sigma}{1-\sigma}} \varphi_{mj}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$

much lower abilities than other teams. So, to complete the equilibrium analysis, we will apply a method called the "share function" approach that is systematically analyzed in Cornes and Hartley (2005), by rewriting the second-stage competition as a Tullock contest with J individual players with heterogeneous marginal costs.[11] Cornes and Hartley (2005) considered a J player (individual) Tullock contest with heterogenous constant marginal costs $w_1 \leq w_2 \leq \ldots \leq w_J$, in which player $j = 1, \ldots, J$ exerts effort X_j with $w_j > 0$. Her winning probability is specified by $P_j = \frac{X_j}{\sum_{k=1}^{J} X_k}$, and her payoff is

$$u_j = \frac{X_j}{\sum_{k=1}^{J} X_k} V - w_j X_j.$$

The payoff function is strictly concave in X_j, and the first-order condition is

$$\frac{\sum_{k \neq j} X_k}{\left(\sum_{k=1}^{J} X_k\right)^2} V - w_j = \frac{X_{-j}}{X^2} V - w_j = 0, \tag{6}$$

for $j = 1, \ldots, J$. Then, $X_j > 0$ is a unique best response to X_{-j} if and only if

$$X_j^2 + 2X_{-j}X_j + X_{-j}^2 - \frac{X_{-j}}{w_j} V = 0.$$

Noting that some players may have too high a marginal cost for an interior solution, player j's best response to X_{-j} is

$${}_j(X_{-j}) = \max \left\{ -X_{-j} + \sqrt{X_{-j}V} \right.$$

X_j is X_j

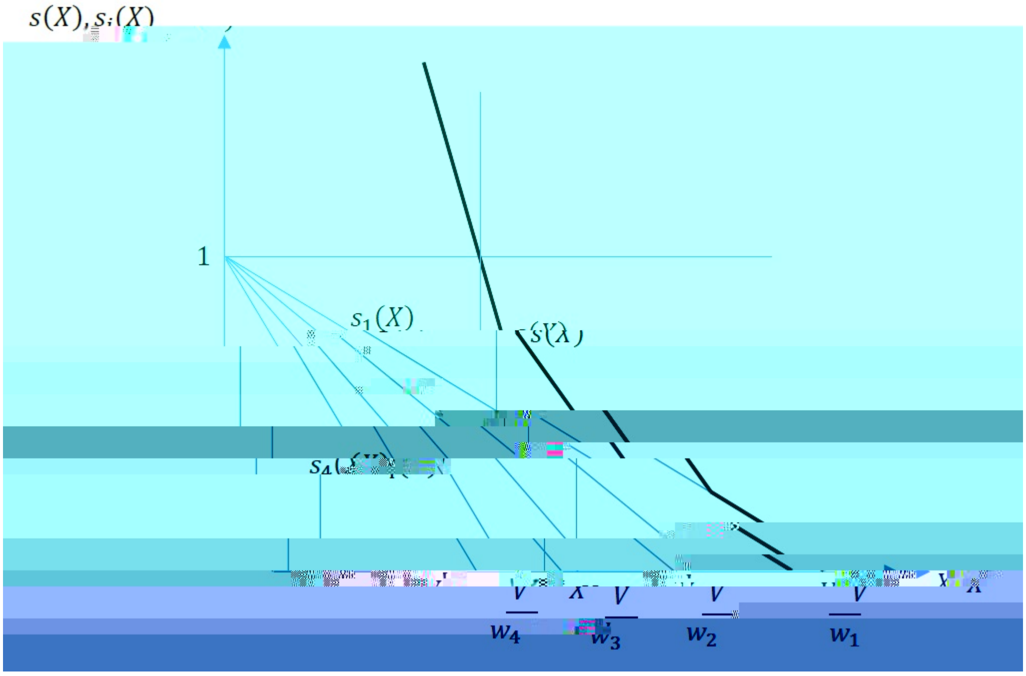

Figure 1: An example with $J = 4$ and $w_1 < w_2 < w_3 < w_4$. Teams 1, 2, and 3 are active. Team 4 is inactive.

$j = 1, \ldots, J$ and $s(X)$. The equilibrium for the artificial contest is a total effort, X^*, for which every group's optimal share sums up to 1. Clearly, there exists a *unique equilibrium* X^* defined by $\sum_k s_k(X^*) = 1$. Moreover, at the equilibrium X^*, $s_j(X^*)$ is also the winning probability of player j. As is easily seen from Figure 1, if $\hat{X}^{n_j} = \frac{V}{w_j} < X^*$, then $s_j(X^*) = 0$ must hold, which means only those groups with smaller marginal costs are *active, i.e., exert positive efforts.* The following lemma summarizes the result of this Tullock game with heterogeneous marginal costs $(J; (w_j)_{j=1}^J)$.

Lemma 2. [Cornes and Hartley, 2005] *A Tullock game with heterogeneous marginal costs* $(J; (w_j)_{j=1}^J)$ *has a unique equilibrium* X^* *at* $\sum_j s_j(X^*) = 1$. *Moreover, there exists* j^* *such that, for each* $j = 1, \ldots, j^*$, $X_j^* = X^* - \frac{w_j (X^*)^2}{V}$, *and for each* $j = j^* + 1, \ldots, J$, $\hat{X}^{n_j} \leq X^*$ *(or* $\sum_k s_k(\hat{X}^{n_j}) \geq 1$*) and* $X_j^* = 0$ for

Theorem 1. *Given a team profile φ, there exists a unique equilibrium in the inter-team contest for any partition of players $\Pi = \{N_1, \ldots, N_J\}$ characterized by the share function $s(X^*) = 1$. There is $j^* \in \{1, \ldots, J\}$ such that $P_j^* = s_j(X^*) > 0$ (active teams) for all $j \leq j^*$ ($\hat{X}_j > X^*$), while $P_j^* = s_j(X^*) = 0$ (inactive teams) for all $j > j^*$ ($\hat{X}_j \leq X^*$). Then, team j*'s winning probability is

$$P_j^* = 1 - \frac{(j^* - 1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}};$$

player $i \in N_j$ of team $j = 1, \ldots, J$ obtains payoff

$$U_{\varphi_{mj}}^* = \begin{cases} V \times \gamma_m P_j^* \left(1 - \frac{(j^*-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}}\right) \left(\frac{a_{\varphi_{mj}} \theta_m}{A_j(\varphi_j)}\right)^{\frac{\sigma}{1-\sigma}} & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}.$$

Moreover, the equilibrium total efforts are

$$X^* = \frac{j^* - 1}{\sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}};$$

and

$$(j^* - 1)\frac{1}{A_j(\varphi_j)} < \sum_{k=1}^{j^*} \frac{1}{A_k(\varphi_k)}$$

holds for all $j = 1, \ldots, j^*$.[12]

4 Stable Team Structures

In this section, we will consider the stability of a team structure generated by a given φ. We will consider a simple concept of *head-hunting*: given $\varphi_{mj} = i' \in N$, a team j offers this position (m, j) to another player i by replacing the incumbent player φ_{mj} by player i. A head-hunting is *successful* if (i) team j's winning probability improves, and (ii) player i who received the offer get a zero payo (Theorem 1). We have the following result (see Appendix for the proof).

Proposition 3. *Suppose that $'$ is immune to a successful head-hunting of unemployed workers. Then, $a_{i^0} \geq a_i$ for all i^0 2 $E(')$ and all i 2 $UE(')$.*

This proposition implies that if we are concerned about stable allocations, then we can focus on the highest $M \times j$ ability players. Given that the highest ability $M \times j$ players are employed initially, and if a head-hunting of an employed player takes place, a vacant position in the head-hunted team and a newly unemployed player (red by the head-hunting team) are generated. If players are totally myopic, and head-hunting decisions are made based on this resulting team structure, there are successful head-huntings that are unreasonable. The following casual example illustrates the point.

Example 1. *Suppose that there are three two-person teams (pairs) of players. The common sharing rule is egalitarian so that both members of a team get 50% share. Player 1 is the highest ability one, and player 2 is the second, and so on: player 6 is the lowest ability player. Now, consider an assortative matching of the players: a great team (players 1 and 2), a very good team (players 3 and 4), and a poor team (players 5 and 6). In this case, the great team has the highest winning probability, and the very good team has the second highest winning probability. The poor team has little chance to win. If players are myopic, there is a successful head-hunting from this intuitively very stable ability-sorted team structure. The great team may kick out player 2, and head-hunt player 3. In this case, players 2 and 4 are left alone, and there are e ectively only two teams: a semi-great team with players 1 and 3 and a poor team with players 5 and 6. The former team's winning probability jumps up close to one without having a serious rival team. This is a successful head-hunting.*

In the above example, players 2 and 4 were unemployed after head-hunting. However, it is natural to think that these two players form a team in reaction to the head-hunting. Since the newly unemployed worker has the highest ability, it is best for the team with the vacancy to make an o er to the newly unemployed player. Thus, it is natural to assume that when team j head-hunts a player who is currently employed by team k, then team j's red player is employed by team k. Team j expects that team k would hire the player red by j, and decide if this head-hunting is pro table.[13] Formally, we say:

De nition 1. *Let $'$ is a feasible allocation, and assume that $E(') = f1; :::; Mj\ g$ where j is the number of active teams (highest Mj ability players are employed). Consider swapping players $i = '_{mj}$ and $h = '_{\ell k}$ for $j; k \leq j$, and let $'^0 = ('_{j^0})_{j^0=1}^{M}$ be the resulting allocation, where (i) $'^0_{j^0} = '_{j^0}$ for all $j^0 \notin j; k$, (ii) $'^0_j = ('_{1j}; :::; '_{m-1j}; h; '_{m+1j}; :::; '_{Mj})$, and (iii) $'^0_k = ('_1$*

employed player for j if (a) $P_j(\varphi') > P_j(\varphi)$ and (b) $U_h(\varphi') > U_h(\varphi)$. We say that φ is stable if φ has no successful head-hunting of neither employed nor unemployed players.

Remark 3. Note that with the above definition of successful head-hunting, head-hunting team j is better off in the Pareto sense except for the former member i who was asked to go. This is because $A(\varphi'_j) > A(\varphi_j)$ implies all team member's payoff goes up (by Theorem 1). Thus, our successful head-hunting implies that the head-hunting team j *unanimously* accepts player h's taking position m. Alternatively, we can define a successful head-hunting by giving priorities to the team leaders' preferences who simply want to maximize their teams' winning probabilities. If a team leader can assign team members to M positions freely, she assigns them to the positions by their abilities in descending order: $a_{\varphi_{1j}} \geq a_{\varphi_{2j}} \geq \ldots \geq a_{\varphi_{Mj}}$. Starting from any membership profile φ_j, if player h is head-hunted for position m from team k, then she would fire φ_{Mj} instead of φ_{mj} by rearranging players as $\varphi'_{mj} = h$, and $\varphi'_{\tilde{m}j} = \varphi_{\tilde{m}-1j}$ for all $\tilde{m} = m+1, \ldots, M$. In this case, players $\varphi_{mj}, \ldots, \varphi_{M-1j}$ may not be better off by player h's joining the team. We can modify our stability concept by using this definition of a successful head-hunting. Our Propositions 4 and 5 are robust to this modification of the definition of stability.[14]

By Proposition 3, if $E(\varphi) = \{1, \ldots, M_j\}$ holds, then there is no successful head-hunting from $UE(\varphi$

the lowest, and the top J players are assigned to position 1 of each team, then next J players are assigned to position 2 of each team, and so on and so forth. This means that team j is composed of players of abilities a_j, a_{j+J}, ..., $a_{j+(M-1)J}$ for all $j = 1, \ldots, J$. In an interesting coalition formation game, Morelli and Park (2016) showed this allocation to be group-stable. If $_m$s are heterogeneous enough, we can show that a cyclical assignment of players over J teams is a stable team structure. To simplify the exposition, we assume that all J teams are active under the cyclical assignment.

We will consider a special family of s which satis es $_{m+1} = \quad _m$ for all $m = 1, \ldots, M-1$ for $\ \in [0, 1]$. We may call this rule a **hierarchical sharing rule**. Let $\ : [0, 1] \to \ ^M$ be such that

$$_m(\ ) = \frac{\ ^{m-1}}{1 + \ + \ldots + \ ^{M-1}} = \frac{(1 - \ )\ ^{m-1}}{1 - \ ^M}$$

for all $m = 1, \ldots, M$. If $\ = 0$, $_1 = 1$ with $_m = 0$ for all $m = 2, \ldots, M$, which is a monopolization rule, while if $\ = 1$ then it is the egalitarian rule $_m = \frac{1}{M}$ for all $m = 1, \ldots, M$. The next proposition shows that the hierarchical sharing rule supports the cyclical assignment allocation for small enough (see Appendix for the proof).

Proposition 5. *Consider hierarchical sharing rules. There is* $\ \in (0, 1)$ *such that for all* $\ \in [0, \ ) >$

$_1 = 0.5$, $_2 = 0.2$, and $_3 = \ _4 = \ _5 = 0.1$. There is a stable allocation $'_1 = (1, 4, 7, 8, 9)$, $'_2 = (2, 5, 10, 11, 12)$, and $'_3 = (3, 6, 13, 14, 15)$, which is a combination of cyclical assignment and ability sorting allocations. Their winning probabilities are: $P_1 = 0.369$, $P_2 = 0.332$, and $P_3 = 0.299$. This sharing rule assigns hierarchical shares but treats lower ranks equally. The resulting allocation reveals that high ability players are spread over teams while low ability players are ability sorted across teams. This pattern may mimic corporates' worker ability distributions.

Finally, we illustrate how our results can be extended to the case with di erent categories of positions and di erent skill types of workers. So far, we assumed that all players belong to the same category and that teams' positions are all symmetric. However, teams may have di erent categories of positions, and players may have di erent skill sets.[15] For exampleti93 To61umay have di

team contests, although ex post each contest is played by a set of di erent teams. Thus, each player's expected payo is calculated as a weighted expected payo of each possible draw of team pro le in the contests | they are playing contests with a distribution of team types in their league.[17] In the team formation stage, each player decides which position is available for her to take based on her expected payo comparison, and a potential team manager can enter the market by o ering a nonexisting sharing rule in the market if possible. This is so much stronger equilibrium concept, and the resulting allocation is strongly stable. We are planning to explore the properties of this free entry equilibrium in such large replica contests.

Appendix

We collect most proofs here.

Proof of Proposition 2. We compute the equilibrium e ort level rst. Recalling (2), we obtain

$$
\begin{aligned}
e_{\varphi_{mj}} &= \left[X_j (1-P_j)\frac{V}{X}\right]^{\frac{1}{1-\sigma}} \left[a^{\sigma}_{\varphi_{mj}}\, m\right]^{\frac{1}{1-\sigma}} \\
&= \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\} \frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} \left[\frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} \frac{V}{\frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}}\right]^{\frac{1}{1-\sigma}} \left[a^{\sigma}_{\varphi_{mj}}\, m\right]^{\frac{1}{1-\sigma}} \\
&= \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\} \left[\frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} \frac{1}{A_j(\varphi_j)}\right]^{\frac{1}{1-\sigma}} \left[a^{\sigma}_{\varphi_{mj}}\, m\right]^{\frac{1}{1-\sigma}}
\end{aligned}
$$

This implies that player i's payo is written as

$$
\begin{aligned}
U_{\varphi_{mj}} &= P_j\, mV - e_{\varphi_{mj}} \\
&= \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\} mV - \left[\left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\} \frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} \frac{1}{A_j(\varphi_j)}\right]^{\frac{1}{1-\sigma}} \left[a^{\sigma}_{\varphi_{mj}}\, m\right]^{\frac{1}{1-\sigma}} \\
&= \left\{1 - \frac{(J-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}}\right\} \left[mV - \left(\frac{(J-1)V}{\sum_{k=1}^{J}\frac{1}{A_k(\varphi_k)}} \frac{1}{A_j(\varphi_j)}\right)^{\frac{1}{1-\sigma}} \left(a^{\sigma}_{\varphi_{mj}}\, m\right)^{\frac{1}{1-\sigma}}\right] \\
&= \left\{1 - \right.
\end{aligned}
$$

Next, we present a useful lemma for the following proofs.

Lemma A1. *Suppose that team $j = 1, 2, \ldots, j^*$ are active, and team $j = j^* + 1, \ldots, J$ are inactive under assignment φ. Let $k \geq j^* + 1$ and $h \leq j^*$, then we have*

$$\frac{j^* - 2}{\sum_{j' \neq k,h}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^* - 1}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^*}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)}}.$$

Proof of Lemma A1. By Theorem 1, since team k is inactive we have

$$\frac{j^* - 1}{A_k(\varphi_k)} \geq \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}.$$

Then

$$\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)} \leq \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{j^* - 1} \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} = \frac{j^*}{j^* - 1} \sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}.$$

Rearranging the above inequality yields $\frac{j^* - 1}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j^*}{\sum_{j'=1}^{j^*} \frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_k(\varphi_k)}}$. The remaining part can be proved in a similar way.

Proof of Proposition 3. Suppose not. Then, there are $\varphi_{mj} \in E(\varphi)$ and $i \in UE(\varphi)$ such that $a_{\varphi_{mj}} < a_i$. Let the number of active teams under φ be j^*. From Proposition 1, we know $P_j = 1 - \frac{(j^* - 1)\frac{1}{A_j(\theta_j)}}{\sum_{k=1}^{j^*} \frac{1}{A_k(\theta_k)}}$ and $A_j(\varphi_j) = \left(\sum_{m'=1}^{M} a_{\varphi_{m'j}}^{\frac{\sigma}{1-\sigma}} \gamma_{m'j}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$. Thus, if team j replaces φ_{mj} by i, the new membership profile denoted by φ' yields a higher productivity $A_j(\varphi'_j) = \left(\sum_{m'=1}^{M} a_{\varphi'_{m'j}}^{\frac{\sigma}{1-\sigma}} \gamma_{m'j}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} > A$

after the swapping and observe that

$$
\begin{aligned}
P_j^{0} &= 1 - \frac{(j-2)\frac{1}{A_j(\varphi_j^{0})}}{\sum_{k=1;k\neq j}^{j-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j^{0})}} \\
&> 1 - \frac{(j-2)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1;k\neq j}^{j-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)}} \\
&> 1 - \frac{(j-1)\frac{1}{A_j(\varphi_j)}}{\sum_{k=1;k\neq j}^{j-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)} + \frac{1}{A_j(\varphi_j)}}.
\end{aligned}
$$

The last inequity holds because of $\frac{1}{A_j(\varphi_j)} < \frac{1}{j-2}\left[\sum_{k=1;k\neq j}^{j-1}\frac{1}{A_k(\varphi_k)} + \frac{1}{A_j(\varphi_j)}\right]$ (team j is active under φ) and Lemma A1.

Proof of Lemma 3. First, we consider the case where the swapping does not change the number of active teams j. Note that $A_j(\varphi_j) \geq A_k(\varphi_k)$ means $\tilde{A}_j(\varphi_j) = \sum_{m^0=1}^{M} a_{m^0 j}^{\frac{\sigma}{1-\sigma}} \ {}_{m^0 j}^{\frac{\sigma}{1-\sigma}} \geq \tilde{A}_k(\varphi_k) = \sum_{m^0=1}^{M} a_{m^0 k}^{\frac{\sigma}{1-\sigma}} \ {}_{m^0 k}^{\frac{\sigma}{1-\sigma}}$. Let $\ = \ {}_{mj} = \ `_k$, and let $\tilde{a}_i \equiv a_i^{\frac{\sigma}{1-\sigma}} \ `_k^{\frac{\sigma}{1-\sigma}} = a_i^{\frac{\sigma}{1-\sigma}} \ ^{\frac{\sigma}{1-\sigma}}$ and $\tilde{a}_h = a^{\sigma}$

$\sum_{j'=1}^{j} \frac{1}{A_{j'}(\varphi'_{j'})}$. These two inequalities imply that

$$\frac{(j - 1)\frac{1}{A_k(\varphi_k)}}{\sum_{j'=1}^{j}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{(j - 1)\frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j}\frac{1}{A_{j'}(\varphi_{j'})}} \geq \frac{j \frac{1}{A_j(\varphi_j)}}{\sum_{j'=1}^{j}\frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_{j+1}(\varphi_{j+1})}}$$

$$> \frac{j \frac{1}{A_j(\varphi'_j)}}{\sum_{j' \neq j;k}\frac{1}{A_{j'}(\varphi_{j'})} + \frac{1}{A_j(\varphi'_j)} + \frac{1}{A_k(\varphi'_k)} + \frac{1}{A_{j+1}(\varphi_{j+1})}}$$

Since $A_j(\varphi_j) = \left(\sum_{\ell=1}^{M} a_{j+(\ell-1)J}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$, $A_k(\varphi'_k) < \left(\left(\frac{1}{\nu}a_{mJ+k}\right)^{\frac{\sigma}{1-\sigma}} + \sum_{\ell \neq m+1}^{M} a_{k+(\ell-1)J}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} < \frac{1}{\nu}A_k(\varphi_k)$. Similarly, $A_j(\varphi'_j) > \left(a_i + \sum_{\ell \neq m}^{M} a_{j+(\ell-1)J}^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}} > A_j(\varphi_j)$ hold. Thus, we have

$$U_i(\varphi'; \ ) < \ _{m+1}\left(1-(J-1)\frac{\frac{1}{A_k(\varphi'_k)}}{\sum_{j' \neq k;j}\frac{1}{A_{j'}(\varphi_{j'})}}\right.$$

and

$$u_3^{sort} = \frac{\ ^2}{1 + \quad + \quad ^2} \quad 1 - \frac{2 \ ^2}{1 + \quad + \quad ^2} \qquad 1 - \frac{2 \ ^2 \ ^2}{(1 + \quad + \quad ^2)\,(1 + \quad + \quad ^2)}$$

We need

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