All probabilities are equal, but some probabilities are more equal than others

Christina Letsou[*] Shlomo Naeh[†] Uzi Segal[‡]

Abstract

A common procedure for selecting people is to have them draw balls from an urn in turn. Modern and ancient stories suggest that such lotteries may be viewed by the individuals as "unfair." We compare this procedure with several alternatives. They all give individuals equal chance of being selected, but have different structures. We analyze these procedures as multistage lotteries. In line with previous literature, our analysis is based on the observation that multistage lotteries are not considered indifferent to their probabilistic one-stage representations. We use a non-expected utility model and show that individuals have preferences over the different procedures.

Key words: Fair lotteries, non-expected utility, multi-stage lotteries
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Introduction

Thirty French hostages in a German prison in occupied France need to select three of them to be executed by their captives in retribution to the killing of three Germans by the Resistance. They tear down an old letter draw them out of a shoe (Graham Greene, *The Tenth Man*). Nine Greek heroes want to duel Hector. Old Nestor suggests a lottery. They mark their lots and cast them in a helmet. Nestor shakes the helmet, and out falls the lot of Ajax (*Iliad* VII, 171–182). Moses needs to select seventy out of seventy-two elders to help him in leading the Israelites in the desert. He intents to mark seventy slips "elder," to put them with two empty slips into an urn, and to ask each of the candidates, in their turn, to draw a lot. To abate a possible claim of unfairness by the elders, he actually marks seventy-two slips "elder" and put them, together with two blank slips into the urn (*Talmud Yerushalmi*, Sanhedrin 1:7).[1] Five crackers are put in a bowl full of bran. In four of them there is a check for 2 million SF, one contains a bomb, strong enough to kill the person who pulls the cracker. Mrs. Montgomery and Mr. Belmont reach the bowl together. "Mrs. Montgomery... crying 'Ladies first', knocked off the lid and plunged her hand into the bran. Perhaps she had calculated that the odds would never be as favorable again. Belmont had probably been thinking along the same lines, for he protested, 'We should have drawn for turns'." (Graham Greene, *Dr Fischer of Geneva or the Bomb Party*).

Each of these procedures needs to select some members of a given group, either for a good or for a bad outcome. In all cases a random tool is used which gives all candidates the same probability of selection. But are these procedures all the same? Clearly, the candidates have preferences over the different mechanisms. Are they wrong?

Statistically — for sure. But this doesn't mean that candidates may not have preferences over the way these probabilities are created. Most of these procedures require more than one stage of randomization, and following the empirical and theoretical literature, we show that the different structures lead to non-indifference between them. Moreover, we show that the intuitive preferences (as reflected by the above and other examples) are derived from the same conditions leading to some violations of expected u one color and one ball of another color are put into a box and in their turn, subjects pick a random ball with no replacement. The selected person is the one who picked the odd ball (similarly to the prison example). 2. Names lottery, where the n names are put in a box and one (or n – 1) are randomly selected (the *Iliad* example). 3. A winning ball is added to the pre-ordered procedure, and if too many people are selected, the procedure starts over again (a variant of Moses and the elders). We compare these procedures and show how their desirability changes with the size of the group and the identity of each participant.

where f is continuous and strictly increasing, $u(0) = 0$, $f(0) = 0$, and $f(1) = 1$. The value of the lottery L is

$$\text{RD}\left(u^{-1}(\text{RD}(X_1)), q_1; \ldots; u^{-1}(\text{RD}(X\ )), q\ \right) \tag{2}$$

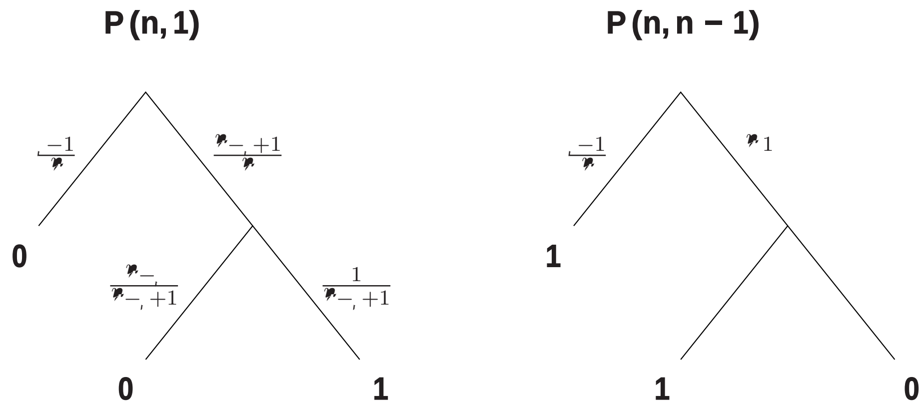
Let $g(p) = 1 - f(1-p)$. Observe that $g(0) = 0, g(1) = 1$, and g is concave i f is convex. Furthermore we can rewrite eq. (1) as

$$u(x_1)g(p_1) + \sum_{=2} u(x\ ) \left[g\left(\sum_{=1} p \right) - g\left(\sum_{=1}^{-1} p \right) \right] \tag{3}$$

We assume throughout that all individuals in society have the same preferences (and will therefore use the same functions u, f, and g to all), that the utility from being selected for a good outcome is 1, and the utility from being selected for a bad outcome is 0. Denote by $v\ (\Gamma)$ the value of procedure Γ to individual i using eq. (2) with either eq. (1) or (3).

The RD model represents risk aversion (in the sense of rejection of mean preserving spreads) i f is convex (and g concave, see Chew, Karni, and Safra [5]). The elasticity of a function $h(p)$ is given by $\eta\ (p) =$ $'($

red. People are pre-ordered, and then one after the other they draw from the


P (n, 1)
P (n, n − 1)
0
0
1
1
1
0

Suppose $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$. Clearly $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$ if and only if $i \leq i^*$. Eq. (6) is less than or equal to zero if and only if

$$\frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) \leq \frac{1}{n} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right)$$

$$\iff \frac{\frac{1}{n-i+1} f'\left(\frac{1}{n-i+1}\right)}{f\left(\frac{1}{n-i+1}\right)} \leq \frac{\frac{n-i+1}{n} f'\left(\frac{n-i+1}{n}\right)}{f\left(\frac{n-i+1}{n}\right)} \tag{7}$$

This inequality holds for $i \leq i^*$ since $\frac{1}{n-i+1} \leq \frac{n-i+1}{n}$ and the elasticity of f is increasing.

On the other hand, suppose $\frac{1}{n-i+1} > \frac{n-i+1}{n}$ which implies that $i > i^*$. Eq. (6) is greater than or equal to zero if and only if the inverse of eq. (7) holds, which holds for $i \geq i^*$ since $\frac{1}{n-i+1} \geq \frac{n-i+1}{n}$ and the elasticity of f is increasing.

The proof for g is similar. ■

The conditions of claim 1 are nonempty. Let $f(p) = \frac{\ p-1}{-1}$. Then $g(p) = \frac{-\ 1-p}{-1}$. It is straightforward to verify that η1

Changing n

When n − 1 out of n people are going to be selected for a good outcome, it seems almost obvious that each of them would like the number n to be as high as possible, since the ex-ante probability of being selected, $n - 1$

person i in $P(n,1)$ becomes person $i+1$ in $P(n+1,1)$. Then

$$\begin{aligned} v_i^{n+1}(P(n+1,1)) &= f\left(\frac{(n+1)-(i+1)+1}{n+1}\right) f\left(\frac{1}{(n+1)-(i+1)+1}\right) \\ &= f\left(\frac{n-i+1}{n+1}\right) f\left(\frac{1}{n-i+1}\right) \\ &< f\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right) = v_i^n(P(n,1)) \end{aligned}$$

Thus, all individuals i for which $i \geq i_0$ are strictly worse off.

Second, consider $i < i_0$. Recall that the value of $P(n,1)$ to person i is given by eq. (4). The derivative of eq. (4) with respect to n is

$$\frac{i-1}{n^2} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right) - \frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right)$$

which is non-positive if and only if

$i-1$

The proof for g is similar. ■

Clearly condition (?) for individual i is satisfied for h(p) = p (observe that the LHS of eq. (8) is less than one and the elasticity of p ≡

Krogh said contemptuously, "Why the quickest way? This is the last gamble some of us will have. We may as well enjoy it."

"The only way is to draw," the mayor said.

The clerk prepared the draw, sacrificing for it one of his le o

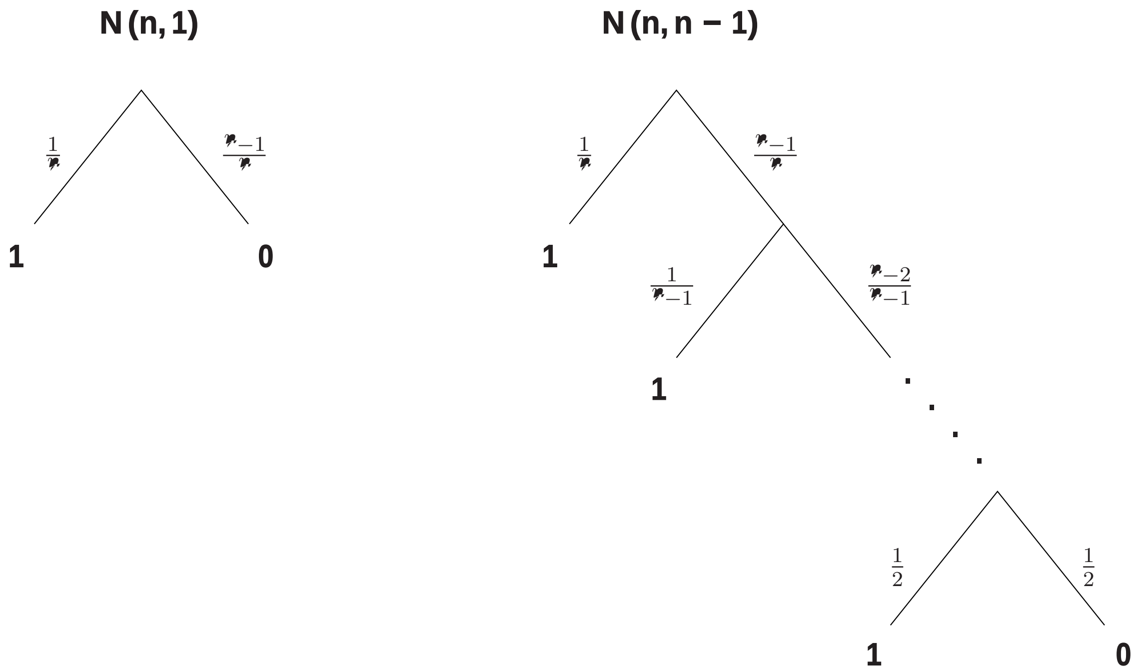


Figure 2: Procedures $N(n,1)$ and $N(n,n-1)$ for each person i

Claim . If the elasticity of f is increasing, then $v_i(P(n,1)) \leq v(N(n,1))$ for all i. If the elasticity of g is decreasing, then $v_i(P(n,n-1)) \geq v(N(n,n-1))$ for all i.

Proof: Since $v(N(n,1)) = f\left(\frac{1}{n}\right)$, the first claim follows immediately by by eq. (5). Observe that, for all i,

$$
\begin{aligned}
v(N(n, n-1)) &= 1 - \prod_{i=1}^{n-1} g\left(\frac{n-i}{n-i+1}\right) \\
&= 1 - \prod_{i=1}^{j-1} g\left(\frac{n-i}{n-i+1}\right) \times \prod_{i=j}^{n-1} g\left(\frac{n-i}{n-i+1}\right) \\
&\leq 1 - g\left(\frac{n-j+1}{n}\right) g\left(\frac{1}{n-j+1}\right) = v_j(P(n, n-1))
\end{aligned}
$$

where the inequality follows by lemma 1. ■

The nine heroes put their lots in a helmet and Ajax's name was drawn. They might as well have drawn eight lots for those who would *not* fight Hector. These two procedures produce the same ex ante probability for being the person to fight, but they are different in one important aspect. The one they used is a single stage lottery. The suggested alternative requires eight stages. Not surprisingly, our analysis does not consider them the same. The value of procedure N_1^n, which is drawing one name out of a hat of n names to win a good outcome is

$$v(N_1^n) = v(N(n, 1)) = f\left(\frac{1}{n}\right) \tag{10}$$

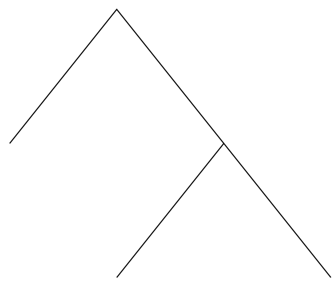
while the value of procedure N_{n-1}, which is drawing n − 1 names out of a hat of n names to receive a bad outcome is

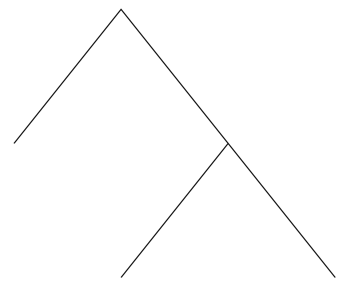
$$v(N_{n-1}) = \prod_{i=1}^{n-1} f\left(\frac{n-i}{n-i+1}\right) \tag{11}$$

procedure is repeated. That is, person i is facing the following multi-stage lottery:

$$W_i(n, n-1) = \left(1, \tfrac{i-1}{n+1}; \left(\left(1, \tfrac{n-i}{n-i+1}; W(n, n-1), \tfrac{1}{n-i+1}\right), \tfrac{n-i+1}{n-i+2}; 0, \tfrac{1}{n-i+2}\right), \tfrac{n-i+2}{n+1}\right)$$

We show next that there is a connection between preferences for adding balls ao





and Yaari [17]) is irrelevant. There is no point in using a "fair" mechanism unless it is deemed fair by those who should bear its consequences. And if adding a green ball to the urn, or having a names lottery rather than sequen-