

You should attempt the problems. Partial credit will be given for serious efforts.

- (1) Let  $S$  be a closed non-orientable surface of genus  $g$ .
  - (a) What is  $H_i(S; \mathbb{Z}_2)$ ? (answer only)
  - (b) Find out the maximal number of disjoint orientation reversing simple closed curves in  $S$ . (Justify your answer)
- (2) Let  $X$  be a path-connected space and  $\tilde{X}$  a universal covering space of  $X$ . Prove that if  $\tilde{X}$  is compact, then  $\pi_1(X)$  is a finite group.
- (3) Let  $M$  be a compact, connected, orientable  $n$ -manifold, where  $n$  is odd.  
(You may assume, if you like, that  $M$  is triangulated.)
  - (a) Show that if  $\partial M = \emptyset$ , then  $\chi(M) = 0$ .
  - (b) Show that if  $\partial M \neq \emptyset$ , then  $\chi(M) = 1$ .

**GT Qual 2011 Part II**  
**Show All Relevant Work!**

1) The image of the map  $X : \mathbf{R}^2 \rightarrow \mathbf{R}^3$  given by

$$X(\theta; \phi) = ((2 + \cos(\phi)) \cos(\theta); (2 + \cos(\phi)) \sin(\theta); \sin(\phi))$$

is the torus obtained by revolving the circle  $(y - 2)^2 + z^2 = 1$  in the  $yz$  plane about the  $z$  axis. Consider the map  $F : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  given by  $F(x; y; z) = (x; z)$  and let  $f = (F$  restricted to the torus).

- a) Compute the Jacobian of the map  $f \circ X$ : (Note that the map  $X$  descends to an embedding of  $S^1 \times S^1$  into  $\mathbf{R}^3$  but we don't need to obsess over the details of this.)
- b) Find all regular values of  $f$ .
- c) Find all level sets of  $f$  that are *not* smooth manifolds (closed embedded sub-manifolds).

2a) Write down the deRham homomorphism for a smooth manifold  $M$ ; explain briefly why this definition is independent of the (two) choices made.

- b) State the deRham Theorem for a smooth manifold  $M$ .
- c) A crucial step in the proof of the deRham Theorem is: If  $M$  is covered by 2 open sets  $U$  and  $V$ , both of which and their intersection satisfy the deRham theorem, then  $M = U \cup V$  satisfies the deRham theorem. Briefly explain how this crucial step is proven.

3a) If  $\omega$  is a differential form, then must it be true that  $d\omega = 0$ ? If yes, then explain your reasoning. If no, then provide a counterexample.

- b) If  $\omega$  and  $\eta$  are closed differential forms, prove that  $\omega \wedge \eta$  is closed.
- c) If, in addition (i.e., continue to assume that  $\omega$  is closed),  $\eta$  is exact, prove that  $\omega \wedge \eta$  is exact.

4) The Chern-Simons form for a hyperbolic 3-manifold with the orthonormal framing  $(E_1; E_2; E_3)$  is the 3-form

$$Q = \left(\frac{1}{8}\right) (!_{12} \wedge !_{13} \wedge !_{23} - !_{12} \wedge \omega_1 \wedge \omega_2 - !_{13} \wedge \omega_1 \wedge \omega_3 - !_{23} \wedge \omega_2 \wedge \omega_3)$$

where  $(\omega_1; \omega_2; \omega_3)$  is the dual co-frame to  $(E_1; E_2; E_3)$  (note that [Lee] uses  $\theta$ , but here we use  $\omega$ ) and the  $!_{ij}$  are the *connection* 1-forms. The connection 1-forms satisfy

$$d\omega_1 = !_{12} \wedge \omega_2 - !_{13} \wedge \omega_3$$

$$d\omega_2 = +!_{12} \wedge \omega_1 - !_{23} \wedge \omega_3$$

$$d\omega_3 = +!_{13} \wedge \omega_1 + !_{23} \wedge \omega_2$$

- a) In  $\mathbf{H}^3 = f(x; y; z) : z > 0$  with the Riemannian metric  $g = \frac{1}{z^2} dx^2 + \frac{1}{z^2} dy^2 + dz^2$ , orthonormalize the framing  $(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}; \frac{\partial}{\partial z})$ :
- b) Compute the associated dual co-frame  $(\omega_1; \omega_2; \omega_3)$ :
- c) For this orthonormal framing (and dual co-frame), in  $(\mathbf{H}^3; g)$ , compute the Chern-